On sound ranging in quasi-metric spaces[1]

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Abstract

We consider the sound ranging, or source localization, problem — determine unknown source point that emitted the wave from known instants when this wave reached known sensors — in quasi-metric spaces, where distance function can be non-symmetric. Under some assumptions, using quasi-metric as a measure of time instead of space, we describe the iterative process with the stopping criterion that approximates the source with any preselected precision. An example, normed spaces with constant wind, is looked into, and we provide the proof-of-concept implementation of the algorithm for $\mathbb{R}_p^m$ with wind in Rust language.

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Introduction

At unknown instant t_0 of time an unknown point $\mathbf{s}$ of space emits a wave, which then propagates through space and reaches the sensors in known points $\{\mathbf{r}_i\}_{i\in I}$ at known instants t_i. To find $\mathbf{s}$ (and t_0) is to solve a *sound ranging* (SR) problem.

SR problem is also called a *source localization*, or *time-difference-of-arrival* (TDOA) problem. To consider it as a mathematical one in a context as general as possible, we ought to define "space", "wave", "propagation" and so on, maybe adding assumptions that do not hold in general case. The majority of researches in this area and corresponding results deal with 3D Euclidean space, with numerous applications in e.g. acoustics [30], geophysics [25], navigation [9], sensor networks [20], surveillance [21], warfare [4]; see [10, Sec. 1] and [16, Sec. 9.1] for more references. There are deviations from "full Euclidicity" though, either because propagation speed is non-constant, or because the underlying space itself is not Euclidean, namely [9], [19], [23].

In [15] we investigated SR in proper metric spaces, and in [14] — in some non-proper ones. Here we generalize the method from [15] with few notions from [14] to some non-symmetric metrics.

We assume that $\mathbf{r}_i$ and t_i are known exactly, although there is a comment on robustness of the proposed method. This is a huge simplification, — dealing with measurement noise is a significant part of solving practice-oriented SR problems ([13, Sec. I, IV, Fig. 1], [16, Ch. 9]).

"♣" denotes an assumption, not true in general case, that we require to hold unless stated otherwise; "•" denotes a well-known statement provided for the sake of completeness without proofs or references (see e.g. [11], [12], [17]).

Preliminaries

Let X be a non-empty set. A function $\tau\colon X\times X\to\mathbb{R}_+$ satisfying the following 2 axioms:

a) $\tau(\mathbf{x},\mathbf{y})=0$ if and only if $\mathbf{x}=\mathbf{y}$,

b) $\forall\mathbf{x},\mathbf{y},\mathbf{z}$: $\tau(\mathbf{x},\mathbf{y})\leqslant\tau(\mathbf{x},\mathbf{z})+\tau(\mathbf{z},\mathbf{y})$ (oriented triangle inequality)

is called a *quasi-metric* [11, Sec. 1.1, p. 5]. If, additionally, $\forall\mathbf{x},\mathbf{y}$: $\tau(\mathbf{x},\mathbf{y})=\tau(\mathbf{y},\mathbf{x})$ (symmetry), then τ is a usual metric. That is, any metric is a quasi-metric.

The pair (X,τ) or just X when τ is fixed is called a *quasi-metric space*. Sometimes this term has another although resembling meaning, cf. [2, Sec. 2.1] or [26, Sec. 1].

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In SR context, quasi-metrics arise naturally when τ means duration in time rather than distance in space. Consider, for example, two bells in two towns, $\mathbf{a}$ and $\mathbf{b}$, and suppose that strong wind blows from $\mathbf{a}$ to $\mathbf{b}$. If $\tau(\mathbf{x},\mathbf{y})$ denotes the time required for the sound of the bell that rings at $\mathbf{x}$ to propagate to $\mathbf{y}$, then clearly $\tau(\mathbf{a},\mathbf{b}) < \tau(\mathbf{b},\mathbf{a})$.

This example introduces the "$\tau(\text{from},\text{to})$" semantics that we use in SR context.

Well-known ways to "symmetrize" a quasi-metric τ and obtain usual metric ρ are $\rho_{\max}(\mathbf{x},\mathbf{y}) = \max\{\tau(\mathbf{x},\mathbf{y}),\tau(\mathbf{y},\mathbf{x})\}$, $\rho_{\min}(\mathbf{x},\mathbf{y}) = \min\{\tau(\mathbf{x},\mathbf{y}),\tau(\mathbf{y},\mathbf{x})\}$, and $\rho_\alpha(\mathbf{x},\mathbf{y}) = \alpha\big(\tau(\mathbf{x},\mathbf{y})+\tau(\mathbf{y},\mathbf{x})\big)$, where $\alpha>0$; moreover, these metrics are equivalent [11]. In particular, we shall use $\rho_{\max}$.

As a direct consequence of $\triangle$-inequality, we have *2nd $\triangle$-inequalities for quasi-metric*:

• $\forall \mathbf{x},\mathbf{y},\mathbf{z}\in X$: $\begin{cases} \big|\tau(\mathbf{x},\mathbf{y})-\tau(\mathbf{z},\mathbf{y})\big| \leqslant \max\big\{\tau(\mathbf{x},\mathbf{z}),\tau(\mathbf{z},\mathbf{x})\big\},\\ \big|\tau(\mathbf{x},\mathbf{y})-\tau(\mathbf{x},\mathbf{z})\big| \leqslant \max\big\{\tau(\mathbf{y},\mathbf{z}),\tau(\mathbf{z},\mathbf{y})\big\}. \end{cases}$

For any $\mathbf{x}\in X$ and $r>0$, the *symmballs with center* $\mathbf{x}$ *of radius* r

$$\widetilde{B}(\mathbf{x},r) = \big\{\mathbf{y}\in X \mid \tau(\mathbf{x},\mathbf{y})<r \ \text{ and } \ \tau(\mathbf{y},\mathbf{x})<r\big\}$$
$$\widetilde{B}[\mathbf{x},r] = \big\{\mathbf{y}\in X \mid \tau(\mathbf{x},\mathbf{y})\leqslant r \ \text{ and } \ \tau(\mathbf{y},\mathbf{x})\leqslant r\big\}$$

are analogues of open ($B(\ldots)$) and closed ($B[\ldots]$) balls of metric spaces; in fact, $\widetilde{B}(\ldots)$ and $\widetilde{B}[\ldots]$ *are* balls in the metric $\rho_{\max}$.

Definition 1. *We say that* (X,τ) has CZS property/is CZS *if*

$$\forall\{\mathbf{x}_n\}_{n\in\mathbb{N}}\subseteq X,\ \forall\{\mathbf{y}_n\}_{n\in\mathbb{N}}\subseteq X\colon\ \tau(\mathbf{x}_n,\mathbf{y}_n)\xrightarrow[n\to\infty]{}0 \Leftrightarrow \tau(\mathbf{y}_n,\mathbf{x}_n)\xrightarrow[n\to\infty]{}0$$

(CZS stands for "Convergence-to-Zero-is-Symmetric".) In particular, in such spaces we can write "$\mathbf{x}_n \xrightarrow[n\to\infty]{} \mathbf{y}$" without specifying either $\tau(\mathbf{x}_n,\mathbf{y})\xrightarrow[n\to\infty]{}0$ or $\tau(\mathbf{y},\mathbf{x}_n)\xrightarrow[n\to\infty]{}0$, since these become equivalent. Obviously, every metric space is CZS.

Example 1. Let $X=\mathbb{R}$ and $\tau_1(\mathbf{x},\mathbf{y}) = \begin{cases}\mathbf{x}-\mathbf{y}, & \mathbf{x}\geqslant\mathbf{y},\\ 1, & \mathbf{x}<\mathbf{y},\end{cases}$ $\qquad \tau_2(\mathbf{x},\mathbf{y}) = \begin{cases}\alpha(\mathbf{x}-\mathbf{y}), & \mathbf{x}\geqslant\mathbf{y},\\ \beta(\mathbf{y}-\mathbf{x}), & \mathbf{x}<\mathbf{y},\end{cases}$

where $\alpha,\beta>0$.

$\lhd$ (X,τ_1) — *Sorgenfrey's quasi-metric* [11, p. 217] — is not CZS, (X,τ_2) is CZS. $\rhd$

Quasi-metric spaces that have CZS property seem to be more suitable models for "physical" spaces, e.g. in our example with bells and wind.

Definition 2. *We say that* (X,τ) has MUB property/is MUB *if*

$$\exists C\in[1;+\infty)\colon\ \forall\mathbf{x},\mathbf{y}\in X\colon\ \tau(\mathbf{x},\mathbf{y})\leqslant C\tau(\mathbf{y},\mathbf{x})$$

("Mirrored-is-Uniformly-Bounded".) It is evident that MUB implies $\tau(\mathbf{x},\mathbf{y})\geqslant\frac{1}{C}\tau(\mathbf{y},\mathbf{x})$ and CZS.

Proposition 1. *(Continuity of CSZ-quasi-metric) Let* (X,τ) *be CZS,* $\mathbf{x}_n\xrightarrow[n\to\infty]{}\mathbf{x}$, $\mathbf{y}_n\xrightarrow[n\to\infty]{}\mathbf{y}$. *Then* $\tau(\mathbf{x}_n,\mathbf{y}_n)\xrightarrow[n\to\infty]{}\tau(\mathbf{x},\mathbf{y})$.

Proof. $\big|\tau(\mathbf{x}_n,\mathbf{y}_n)-\tau(\mathbf{x},\mathbf{y})\big| \leqslant \big|\tau(\mathbf{x}_n,\mathbf{y}_n)-\tau(\mathbf{x},\mathbf{y}_n)\big| + \big|\tau(\mathbf{x},\mathbf{y}_n)-\tau(\mathbf{x},\mathbf{y})\big| \leqslant$
$$\leqslant \max\big\{\tau(\mathbf{x}_n,\mathbf{x}),\tau(\mathbf{x},\mathbf{x}_n)\big\} + \max\big\{\tau(\mathbf{y}_n,\mathbf{y}),\tau(\mathbf{y},\mathbf{y}_n)\big\} \xrightarrow[n\to\infty]{} 0$$ □

When (X,τ) has CZS property,

1) $A\subseteq X$ is called *relatively compact* if $\forall\{\mathbf{x}_n\}_{n\in\mathbb{N}}\subseteq A$: $\exists\{\mathbf{x}_{n_k}\}_{k\in\mathbb{N}}$, $\exists\mathbf{x}\in X$: $\mathbf{x}_{n_k}\xrightarrow[k\to\infty]{}\mathbf{x}$.

2) $A\subseteq X$ is called *compact* if $\forall\{\mathbf{x}_n\}_{n\in\mathbb{N}}\subseteq A$: $\exists\{\mathbf{x}_{n_k}\}_{k\in\mathbb{N}}$, $\exists\mathbf{x}\in A$: $\mathbf{x}_{n_k}\xrightarrow[k\to\infty]{}\mathbf{x}$.

More on quasi-metrics and their examples can be found in [11], [31].

♣0. If $\mathbf{x}\in X$ emits the sound at the instant $t_0\in\mathbb{R}$ of time, then at any instant $t\geqslant t_0$ the (front of) sound wave is $\big\{\mathbf{y}\in X \mid \tau(\mathbf{x},\mathbf{y})=t-t_0\big\}$. Before t_0, the wave is $\varnothing$.

Here, again, $\tau(\mathbf{x}, \mathbf{y})$ is how long the wave emitted in $\mathbf{x}$ propagates to $\mathbf{y}$.

Note that 1) in this model, the wave passes through any point only once (no reverberations, no "round-the-world" echoes), and 2) the speed of sound is implicit.

1 SR in quasi-metric spaces

We denote the sound source by $\mathbf{s} \in X$ and the emission instant by $t_0 \in \mathbb{R}$; both are unknown.

The sensors $R = \{\mathbf{r}_i\}_{i \in I} \subseteq X$ and the instants $\{t_i\}_{i \in I}$, $t_i = t_0 + \tau(\mathbf{s}, \mathbf{r}_i)$ when propagating sound wave reaches them are known.

$\big\{(\mathbf{r}_i, t_i)\big\}_{i \in I}$ defines the SR problem (SRP). A pair $(\mathbf{s}', t')$ that satisfies

$$\forall i \in I\colon\ t_i = t' + \tau(\mathbf{s}', \mathbf{r}_i)$$

is called a *solution* of this SRP. We are more interested to obtain $\mathbf{s}'$ than t', the latter is trivially obtained from any equation by simply substituting the former.

♣1. Sensors are distinct: $i \neq j \Rightarrow \mathbf{r}_i \neq \mathbf{r}_j$.

Therefore $|R| = |I|$. This assumption is not necessary, but lessens redundancy.

♣2. SRP solution is unique.

One clear way to violate this assumption is to have "not enough" sensors, e.g. much less than "dimensionality" of X. The other way is inappropriate placement of sensors, which depends on X as well. Consider $X = X_1 \cup X_2$, where $X_1 \cap X_2 = \{\mathbf{c}\}$ (two big rooms connected with tiny hole in the common wall). Let $R \subseteq X_1$ and $\mathbf{s} \in X_2$. Then, regardless of $|R|$, from "point of view" of all $\mathbf{r}_i$, $\mathbf{s}' = \mathbf{c}$ at the instant $t' = t_0 + \tau(\mathbf{s}, \mathbf{c})$ is a SRP solution; moreover, every $\mathbf{s}'' \in X_2$ at $t'' = t' - \tau(\mathbf{s}'', \mathbf{c})$ is, — the "structure" of X_2, reflected in SR data, "collapses" when the wave passes through $\mathbf{c}$. This example is somewhat related to Huygens' principle [5, §1].

Definition 3. *For any* $\mathbf{x} \in X$ *and any* $i \in I$, *the* backward instant

$$T_i(\mathbf{x}) = t_i - \tau(\mathbf{x}, \mathbf{r}_i)$$

is the (unique) instant when $\mathbf{x}$ has to emit the wave so that it reaches $\mathbf{r}_i$ at the instant t_i. In particular, $T_i(\mathbf{s}) \equiv t_0$, while for other points they are different. And we are going to use the measure of their "spread" to approximate $\mathbf{s}$.

Definition 4. *For any* $\mathbf{x} \in X$, *the* defect

$$D(\mathbf{x}) = \sup_{i,j \in I} \big|T_i(\mathbf{x}) - T_j(\mathbf{x})\big| = \sup_i T_i(\mathbf{x}) - \inf_i T_i(\mathbf{x})$$

♣2 implies

Proposition 2. $D(\mathbf{x}) = 0$ *if and only if* $\mathbf{x} = \mathbf{s}$.

Remark. In general case, a defect can have local minima besides global minimum $\mathbf{s}$, even if measurements are exact. [15, Ex. 2.1, 2.2] provides few examples for the slightly different defect.

Proposition 3. *For any* $\mathbf{x} \in X$

$$D(\mathbf{x}) \leqslant 2 \max\big\{\tau(\mathbf{s}, \mathbf{x}), \tau(\mathbf{x}, \mathbf{s})\big\}$$

Proof. By 2nd $\triangle$-inequality, $\forall i, j \in I$:

$$\begin{aligned} \big|T_i(\mathbf{x}) - T_j(\mathbf{x})\big| &= \big|t_0 + \tau(\mathbf{s}, \mathbf{r}_i) - \tau(\mathbf{x}, \mathbf{r}_i) - t_0 - \tau(\mathbf{s}, \mathbf{r}_j) + \tau(\mathbf{x}, \mathbf{r}_j)\big| \leqslant \\ &\leqslant \big|\tau(\mathbf{s}, \mathbf{r}_i) - \tau(\mathbf{x}, \mathbf{r}_i)\big| + \big|\tau(\mathbf{s}, \mathbf{r}_j) - \tau(\mathbf{x}, \mathbf{r}_j)\big| \leqslant 2 \max\big\{\tau(\mathbf{s}, \mathbf{x}), \tau(\mathbf{x}, \mathbf{s})\big\} \end{aligned}$$

which implies the sought inequality. □

Corollary 1. $\forall \mathbf{x} \in X\colon D(\mathbf{x}) < +\infty$.

Corollary 2. *If* $\mathbf{s} \in \widetilde{B}[\mathbf{x}, r]$*, then* $D(\mathbf{x}) \leqslant 2r$*.*

Corollary 3. *If* $D(\mathbf{x}) > 2r$*, then* $\mathbf{s} \notin \widetilde{B}[\mathbf{x}, r]$*.*

Test for a symmball. All symmballs $\widetilde{B}[\mathbf{x}, r]$ of (X, τ) are divided into 2 families:
$\mathcal{N} = \{\widetilde{B}[\mathbf{x}, r] \mid D(\mathbf{x}) > 2r\}$ and $\mathcal{S} = \{\widetilde{B}[\mathbf{x}, r] \mid D(\mathbf{x}) \leqslant 2r\}$
Due to Cor. 3, any symmball from $\mathcal{N}$ ("negative") does not contain $\mathbf{s}$. A ball from $\mathcal{S}$ ("suspicious") may or may not contain $\mathbf{s}$.

Proposition 4. *For any* $\mathbf{x}, \mathbf{y} \in X$

$$\big|D(\mathbf{x}) - D(\mathbf{y})\big| \leqslant 2 \max\big\{\tau(\mathbf{x}, \mathbf{y}), \tau(\mathbf{y}, \mathbf{x})\big\}$$

Proof. $D(\mathbf{x}) = \|\sigma(\mathbf{x})\|_\infty$, where $\sigma(\mathbf{x}) = \{T_i(\mathbf{x}) - T_j(\mathbf{x})\}_{(i,j) \in I \times I} \in \mathbb{R}_\infty^{|I \times I|}$. Thus the 2nd $\triangle$-inequality for a norm, $\big|\|\sigma_1\| - \|\sigma_2\|\big| \leqslant \|\sigma_1 - \sigma_2\|$, implies

$$\big|D(\mathbf{x}) - D(\mathbf{y})\big| \leqslant \sup_{i,j} \big|\big(T_i(\mathbf{x}) - T_j(\mathbf{x})\big) - \big(T_i(\mathbf{y}) - T_j(\mathbf{y})\big)\big|$$

and it follows from the $\triangle$-inequality for a norm that

$$\big|D(\mathbf{x}) - D(\mathbf{y})\big| \leqslant \sup_i \big|T_i(\mathbf{x}) - T_i(\mathbf{y})\big| + \sup_j \big|T_j(\mathbf{x}) - T_j(\mathbf{y})\big| = 2 \sup_i \big|T_i(\mathbf{x}) - T_i(\mathbf{y})\big|$$

$\forall i \in I$ we have $\big|T_i(\mathbf{x}) - T_i(\mathbf{y})\big| = \big|t_i - \tau(\mathbf{x}, \mathbf{r}_i) - t_i + \tau(\mathbf{y}, \mathbf{r}_i)\big| \leqslant \max\{\tau(\mathbf{x}, \mathbf{y}), \tau(\mathbf{y}, \mathbf{x})\}$, hence the statement of the proposition holds. □

Definition 5. *We say that* $A \subseteq X$ *has* SDN *property/is* SDN *("if-Small-Defect-then-Near") if*

$$\forall \delta > 0\text{:} \quad \inf_{\mathbf{x} \in A \setminus \widetilde{B}(\mathbf{s}, \delta)} D(\mathbf{x}) > 0$$

($\inf \varnothing = +\infty$.) Put differently, if A is SDN, then $\forall \delta > 0 \ \exists \varepsilon \big(= \inf_{\mathbf{x} \in A \setminus \widetilde{B}(\mathbf{s}, \delta)} D(\mathbf{x})\big) > 0$ such that if $\mathbf{x} \in A$ and $D(\mathbf{x}) < \varepsilon$, then, by definition, $\mathbf{x} \in \widetilde{B}(\mathbf{s}, \delta)$.

If A is SDN and $B \subseteq A$, then B is SDN too.

Proposition 5. *If* (X, τ) *satisfies CZS and* $A \subseteq X$ *is relatively compact, then* A *is SDN.*

Proof. Assume the contrary for some $\delta_0 > 0$ and let $\{\mathbf{x}_n\}_{n \in \mathbb{N}} \subseteq A \setminus \widetilde{B}(\mathbf{s}, \delta_0)$ be such that $\forall n \in \mathbb{N}$: $D(\mathbf{x}_n) < \frac{1}{n}$. A is relatively compact, so $\exists \{\mathbf{x}_{n_k}\}_{k \in \mathbb{N}}$ and $\exists \mathbf{x} \in X$: $\mathbf{x}_{n_k} \xrightarrow[k \to \infty]{} \mathbf{x}$. Prop. 4 implies $\big|D(\mathbf{x}_{n_k}) - D(\mathbf{x})\big| \xrightarrow[k \to \infty]{} 0$, and, since $D(\mathbf{x}_{n_k}) \xrightarrow[k \to \infty]{} 0$ as well, we have $D(\mathbf{x}) = 0$. Thus (Prop. 2) $\mathbf{x} = \mathbf{s}$, which contradicts $\mathbf{x}_{n_k} \notin \widetilde{B}(\mathbf{s}, \delta_0) \Leftrightarrow \max\big\{\tau(\mathbf{s}, \mathbf{x}_{n_k}), \tau(\mathbf{x}_{n_k}, \mathbf{s})\big\} \geqslant \delta_0$. □

Remark. Non-SDN sets and SDN sets that are not relatively compact exist, see [14, p. 4, 8].

Definition 6. *A* coverand $C[\mathbf{a}, r]$ *is a subset of* $\widetilde{B}[\mathbf{a}, r]$ *defined by its* anchor $\mathbf{a} \in X$ *and* radius $r > 0$*, at that* $\mathbf{a} \in C[\mathbf{a}, r]$*.*

Example 2. Consider (X, τ) that is CSZ and *proper* — every $\widetilde{B}[\mathbf{x}, r]$ is compact. Due to Prop. 5, any such symmball is a coverand that has SDN property, with anchor $\mathbf{x}$ and radius r. More general example: discard CSZ of (X, τ), but let $S \subseteq X$ have SDN property. Then $\forall \mathbf{x} \in S$: $C[\mathbf{x}, r] = \widetilde{B}[\mathbf{x}, r] \cap S$ is a coverand with SDN property too.

♣3. For any coverand $C[\mathbf{a}, r]$ we can construct its cover by finite number of coverands of halved radius: $C[\mathbf{a}, r] \subseteq \bigcup_{l=1}^{n} C[\mathbf{a}_l, \frac{r}{2}]$.

♣4. A coverand $C_0 = C[\mathbf{a}_0, r_0] \ni \mathbf{s}$ is known.

♣5. $\exists S \subseteq X$: S is SDN and all coverands constructed in the following process:
0) $\mathcal{B}_0 = \{C_0\}$,
1) $\mathcal{B}_1$ is the set of coverands of radius $\frac{r_0}{2}$ that cover C_0,
...

k) $\mathcal{B}_k$ is the set of coverands of radius $\frac{r_0}{2^k}$ that cover all coverands of radius $\frac{r_0}{2^{k-1}}$ from $\mathcal{B}_{k-1}$ in accordance with ♣3,

...

are subsets of S. That is, $\bigcup\limits_{k=0}^{\infty} \bigcup\limits_{C \in \mathcal{B}_k} C \subseteq S$.

It follows from ♣5 that all coverands we consider below in this section have SDN property.

Now, we approximate $\mathbf{s}$ under ♣0–5.

Refining-Cover-by-Defect (RCD). Beforehand we choose the precision $\delta > 0$.

Step 0. $k := 0$. Let $\mathcal{C}_0 = \{C_0\}$, where C_0 is from ♣4.

Step 1. Due to ♣3, replace each $C[\mathbf{a}, r_k] \in \mathcal{C}_k$ with its cover by $\{C[\mathbf{a}_l, r_{k+1}]\}_l$, where $r_{k+1} = \frac{r_k}{2} = \frac{r_0}{2^{k+1}}$. Test each $C[\mathbf{a}_l, r_{k+1}] = C$: is $D(\mathbf{a}_l) \leqslant 2r_{k+1}$? If yes, add C to $\mathcal{C}_{k+1}$, else discard C.

Since $\mathbf{s} \in \bigcup\limits_{C \in \mathcal{C}_k} C \subseteq \bigcup\limits_{l} C[\mathbf{a}_l, r_{k+1}]$, $\exists l_0$: $\mathbf{s} \in C[\mathbf{a}_{l_0}, r_{k+1}] \subseteq \widetilde{B}[\mathbf{a}_{l_0}, r_{k+1}]$. By Cor. 2, coverand $C[\mathbf{a}_{l_0}, r_{k+1}]$ passes the test, so $\mathcal{C}_{k+1} \neq \varnothing$ at the end of this step. On the other hand, $|\mathcal{C}_{k+1}| \in \mathbb{N}$.

Step 2. Let $\mathbf{z}_{k+1}$ be the anchor of the arbitrarily chosen coverand from $\mathcal{C}_{k+1}$.

Step 3. $k := k+1$. Let $d_k = r_k + \max\limits_{C[\mathbf{a}, r_k] \in \mathcal{C}_k} \max\{\tau(\mathbf{a}, \mathbf{z}_k), \tau(\mathbf{z}_k, \mathbf{a})\}$. If $d_k < \delta$, then halt, else goto Step 1.

Cf. e.g. "Multilevel sample algorithm" [18, Sec. 4] and "Matched Field Processing" [8] in general. Although the approach to localize the source is not TDOA there, they apply "had the source been that point, how much would have sensor data differed from real data?" and "use that difference to compare potential positions of the source" notions. In the former, there is also "discard bad points and refine the mesh around good ones" notion. Note that the "difference" applies to "the present of sensors" — to effect, while in RCD we use "spread" of "the past(s) of source" — of cause, in time. Cf. also [1, Sec. VI].

Proposition 6. *The execution of RCD algorithm eventually halts, at that* $\mathbf{s} \in \widetilde{B}(\mathbf{z}_k, \delta)$.

Proof. Obviously, $r_k = \frac{r_0}{2^k} \xrightarrow[k\to\infty]{} 0$.

Take any $\gamma > 0$. $\mathcal{C}_k \subseteq \mathcal{B}_k$, ♣5, and SDN property of S imply that $\exists \varepsilon > 0$: if $\mathbf{a} \in S$ and $D(\mathbf{a}) < \varepsilon$, then $\mathbf{a} \in \widetilde{B}(\mathbf{s}, \gamma)$. In particular, as soon as $2r_k < \varepsilon$, $D(\mathbf{a}) \leqslant 2r_k < \varepsilon$ for all anchors $\mathbf{a}$ (and $\mathbf{z}_k$) of coverands from $\mathcal{C}_k$, because they have passed the test. Therefore for all such $k \geqslant k_\varepsilon$

$$\tau(\mathbf{a}, \mathbf{z}_k) \leqslant \tau(\mathbf{a}, \mathbf{s}) + \tau(\mathbf{s}, \mathbf{z}_k) < 2\gamma, \quad \tau(\mathbf{z}_k, \mathbf{a}) \leqslant \tau(\mathbf{z}_k, \mathbf{s}) + \tau(\mathbf{s}, \mathbf{a}) < 2\gamma$$

and $\max\limits_{C[\mathbf{a}, r_k] \in \mathcal{C}_k} \max\{\tau(\mathbf{a}, \mathbf{z}_k), \tau(\mathbf{z}_k, \mathbf{a})\} < 2\gamma$ as well. In other words, $\max\limits_{C[\mathbf{a}, r_k] \in \mathcal{C}_k} \max\{\ldots\} \xrightarrow[k\to\infty]{} 0$.

Thus $d_k \xrightarrow[k\to\infty]{} 0$ too, and the execution halts as soon as $d_k < \delta$. At that let $\widehat{\mathbf{a}}$ be the anchor of a coverand from $\mathcal{C}_k$ that contains $\mathbf{s}$, then

$$\tau(\mathbf{s}, \mathbf{z}_k) \leqslant \tau(\mathbf{s}, \widehat{\mathbf{a}}) + \tau(\widehat{\mathbf{a}}, \mathbf{z}_k) \leqslant r_k + \max_{C[\mathbf{a}, r_k] \in \mathcal{C}_k} \max\{\tau(\mathbf{a}, \mathbf{z}_k), \tau(\mathbf{z}_k, \mathbf{a})\} = d_k < \delta$$

similarly $\tau(\mathbf{z}_k, \mathbf{s}) < \delta$. Hence $\mathbf{s} \in \widetilde{B}(\mathbf{z}_k, \delta)$. □

2 Sound propagation in normed spaces with constant wind

Let $(X, \|\cdot\|)$ be a *normed space* over $\mathbb{R}$, i.e. X is a linear vector space over $\mathbb{R}$ and $\|\cdot\|\colon X \to \mathbb{R}_+$ satisfies 3 axioms of norm. Recall that $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ is the corresponding metric.

We describe the quasi-metric τ, based on $\|\cdot\|$, that takes into account a "wind": constant movement of the whole "atmosphere", defined by velocity $\mathbf{v}$ such that "speed of wind" $\|\mathbf{v}\| < 1$.

Let the "speed of sound" be 1, — sound is faster than wind, thus we expect that the sound emitted at any point reaches any other point sooner or later.

So, let $\tau(\mathbf{x}, \mathbf{y}) = t$ be the time needed for the sound wave emitted at $\mathbf{x}$ to propagate to $\mathbf{y}$. To elaborate this definition, we reason as follows: the atmosphere moved by $t\mathbf{v}$ since the emission, in particular, the point of it that had been $\mathbf{x}$ at that instant moved to $\mathbf{x} + t\mathbf{v}$. Now, t units of time after the emission, the sound wave within the atmosphere is the sphere of radius t with the center at that point. That is, t is the solution of the equation

$$\|\mathbf{x} + t\mathbf{v} - \mathbf{y}\| = t \tag{\#}$$

Properness of this definition, — that such t exists and is unique, — follows e.g. from Fixed Point Theorem [17, Sec. 2.8], applied to the mapping $g\colon \mathbb{R} \to \mathbb{R}$: $g(t) = \|\mathbf{x} - \mathbf{y} + t\mathbf{v}\|$, since $\|g(t') - g(t'')\| \leqslant \|t'\mathbf{v} - t''\mathbf{v}\| = \|\mathbf{v}\| \cdot |t' - t''|$; sought t is the fixed point of $g(\cdot)$. As usual, FPT gives practical method to approximate $\tau(\mathbf{x}, \mathbf{y})$, by iterations $t^{(n+1)} = g(t^{(n)})$, where $t^{(0)}$ is arbitrary. $t^{(n)} \xrightarrow[n\to\infty]{} t$, but we halt as soon as $|t^{(n)} - t^{(n-1)}| < \Delta$, for some small precision Δ.

Proposition 7. *For any* $\mathbf{x}, \mathbf{y} \in X$*:*

1) $\forall\lambda \geqslant 0$*:* $\tau(\lambda\mathbf{x}, \lambda\mathbf{y}) = \lambda\tau(\mathbf{x}, \mathbf{y})$*,*

2) $\forall\mathbf{z} \in X$*:* $\tau(\mathbf{x} + \mathbf{z}, \mathbf{y} + \mathbf{z}) = \tau(\mathbf{x}, \mathbf{y})$*,*

3) $\tau(-\mathbf{x}, -\mathbf{y}) = \tau(\mathbf{y}, \mathbf{x})$*.*

Proof. Let $t = \tau(\mathbf{x}, \mathbf{y})$, by definition $\|\mathbf{x} - \mathbf{y} + t\mathbf{v}\| = t$. To obtain (1), multiply this equality by λ: $\|\lambda\mathbf{x} - \lambda\mathbf{y} + (\lambda t)\mathbf{v}\| = \lambda t$, thus (solution of (#) is unique) $\tau(\lambda\mathbf{x}, \lambda\mathbf{y}) = \lambda t$.

(2) and (3) are even more trivial. □

Proposition 8. *For any* $\mathbf{x}, \mathbf{y} \in X$*:*

$$\tau(\mathbf{x}, \mathbf{y}) \in \left[\tfrac{\|\mathbf{x}-\mathbf{y}\|}{1+\|\mathbf{v}\|}; \tfrac{\|\mathbf{x}-\mathbf{y}\|}{1-\|\mathbf{v}\|}\right]$$

Proof. When $\mathbf{x} = \mathbf{y}$, the statement is trivial; suppose $\mathbf{x} \neq \mathbf{y}$ and let $\mathbf{z} = \mathbf{x} - \mathbf{y}$. By definition above, $\tau(\mathbf{x}, \mathbf{y})$ is $t > 0$ such that $\|t\mathbf{v} + \mathbf{z}\| = t$. Let $u = \frac{1}{t}$, then $\|\mathbf{v} + u\mathbf{z}\| = 1$.

Assuming that $u < \frac{1-\|\mathbf{v}\|}{\|\mathbf{z}\|}$, we have $\|\mathbf{v} + u\mathbf{z}\| \leqslant \|\mathbf{v}\| + u\|\mathbf{z}\| < 1$, — a contradiction.

Assuming that $u > \frac{1+\|\mathbf{v}\|}{\|\mathbf{z}\|}$, we have $\|\mathbf{v} + u\mathbf{z}\| \geqslant \big|u\|\mathbf{z}\| - \|\mathbf{v}\|\big|$, and due to $u\|\mathbf{z}\| - \|\mathbf{v}\| > 1 + \|\mathbf{v}\| - \|\mathbf{v}\| = 1$ we obtain $\|\mathbf{v} + u\mathbf{z}\| > 1$, — a contradiction.

Therefore $u \in \left[\frac{1-\|\mathbf{v}\|}{\|\mathbf{z}\|}; \frac{1+\|\mathbf{v}\|}{\|\mathbf{z}\|}\right]$, so $t \in \left[\frac{\|\mathbf{z}\|}{1+\|\mathbf{v}\|}; \frac{\|\mathbf{z}\|}{1-\|\mathbf{v}\|}\right]$. □

Corollary 4. *For any* $\mathbf{x} \in X$*,* $r > 0$*:*
$B\big(\mathbf{x}, r(1 - \|\mathbf{v}\|)\big) \subseteq \widetilde{B}(\mathbf{x}, r) \subseteq B\big(\mathbf{x}, r(1 + \|\mathbf{v}\|)\big)$, $\quad B\big[\mathbf{x}, r(1 - \|\mathbf{v}\|)\big] \subseteq \widetilde{B}[\mathbf{x}, r] \subseteq B\big[\mathbf{x}, r(1 + \|\mathbf{v}\|)\big]$
where $B(\ldots)$ *are* $B[\ldots]$ *are balls in* (X, ρ)*.*

Corollary 5. (X, τ) *has MUB property and thus has CZS property.*

Proof. Take any distinct $\mathbf{x}, \mathbf{y} \in X$. It follows from Prop. 8 that $\frac{\tau(\mathbf{x},\mathbf{y})}{\tau(\mathbf{y},\mathbf{x})} \in \left[\frac{1-\|\mathbf{v}\|}{1+\|\mathbf{v}\|}; \frac{1+\|\mathbf{v}\|}{1-\|\mathbf{v}\|}\right]$. In particular, $\tau(\mathbf{x}, \mathbf{y}) \leqslant \frac{1+\|\mathbf{v}\|}{1-\|\mathbf{v}\|} \cdot \tau(\mathbf{y}, \mathbf{x})$, which is MUB. □

Corollary 6. *For any* $\{\mathbf{x}_n\}_{n\in\mathbb{N}}, \{\mathbf{y}_n\}_{n\in\mathbb{N}} \subseteq X$*:*

$$\|\mathbf{x}_n - \mathbf{y}_n\| \xrightarrow[n\to\infty]{} 0 \Leftrightarrow \tau(\mathbf{x}_n, \mathbf{y}_n) \xrightarrow[n\to\infty]{} 0$$

In particular, $\mathbf{x}_n \xrightarrow[n\to\infty]{\|\cdot\|} \mathbf{y} \Leftrightarrow \mathbf{x}_n \xrightarrow[n\to\infty]{\tau} \mathbf{y}$*.*

- Any finite-dimensional normed space is proper.

Corollary 7. *If* X *is finite-dimensional, then* (X, τ) *is* proper *in the sense that any closed symmball* $\widetilde{B}[\mathbf{x}, r]$ *is compact in* (X, τ)*.*

Proof. Without loss of generality, we consider the symmball $\widetilde{B} = \widetilde{B}[\mathbf{0}, 1]$. Take any $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \subseteq \widetilde{B}$.

By Cor. 4, $\widetilde{B} \subseteq B = B[\mathbf{0}, 1 + \|\mathbf{v}\|]$, which is compact in $(X, \|\cdot\|)$ due to properness of the latter. Thus $\exists\{\mathbf{x}_{n_k}\}_{k\in\mathbb{N}}$, $\exists\mathbf{x} \in B$: $\|\mathbf{x}_{n_k} - \mathbf{x}\| \xrightarrow[k\to\infty]{} 0$. Cor. 6 implies $\mathbf{x}_{n_k} \xrightarrow[k\to\infty]{\tau} \mathbf{x}$. Prop. 1 implies $\tau(\mathbf{0}, \mathbf{x}) = \lim\limits_{k\to\infty} \tau(\mathbf{0}, \mathbf{x}_{n_k}) \leqslant 1$ and $\tau(\mathbf{x}, \mathbf{0}) = \lim\limits_{k\to\infty} \tau(\mathbf{x}_{n_k}, \mathbf{0}) \leqslant 1$, since $\tau(\mathbf{0}, \mathbf{x}_n) \leqslant 1$ and $\tau(\mathbf{x}_n, \mathbf{0}) \leqslant 1$ are provided by definition of $\widetilde{B}$. That is, $\mathbf{x} \in \widetilde{B}$. □

Conclusion

Speaking of RCD algorithm, nothing particularly new has appeared in comparison with [15], — certain conditions of the form "$\rho(\mathbf{x}, \mathbf{y})$ is such that …" become "$\tau(\mathbf{x}, \mathbf{y})$ *and* $\tau(\mathbf{y}, \mathbf{x})$ are such that …", — which is not surprising, since $\max\{\tau(\mathbf{x}, \mathbf{y}), \tau(\mathbf{y}, \mathbf{x})\}$ is a metric. In a sense, we have reduced the quasi-metric case to the metric one.

We have seen that normed spaces with constant wind naturally originate the quasi-metric related to the time of wave propagation, and such quasi-metric has certain convenient properties.

Future work: try to weaken constraints ♣1–5; optimize the RCD algorithm, perhaps under additional assumptions; continue the generalization of SR context.

Appendix: implementation of RCD for $\mathbb{R}^m_p$ with wind

Let $m \in \mathbb{N}$, $p \in [1; +\infty)$, and $X = \mathbb{R}^m$. We recall that $\mathbb{R}^m_p$ is defined by the norm $\|\mathbf{x}\| = \big(\sum\limits_{j=1}^{m} |x_j|^p\big)^{\frac{1}{p}}$. $p = 2$ is Euclidean case.

Following Sec. 2, we consider the quasi-metric τ of sound propagation, based on this norm and the wind defined by its velocity $\mathbf{v} \in B(\mathbf{0}, 1)$. At that we distinguish balls $B(\ldots)$ and $B[\ldots]$ in norm-induced metric from symmballs $\widetilde{B}(\ldots)$ and $\widetilde{B}[\ldots]$ in τ.

Being given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, we approximate $\tau(\mathbf{x}, \mathbf{y})$ by iterations $t^{(n+1)} = g(t^{(n)})$ with precision Δ.

By Cor. 5, Cor. 7, and following Ex. 2, we can use closed symmballs $\widetilde{B}[\mathbf{a}, r]$ as coverands $C[\mathbf{a}, r]$. Let us construct the refined cover for ♣3. Due to Prop. 7, without loss of generality it suffices to construct such cover by $C[\mathbf{a}_l, \frac{1}{2}]$ for $C[\mathbf{0}, 1]$.

By Cor. 4, $C[\mathbf{0}, 1] \subseteq B[\mathbf{0}, 1 + \|\mathbf{v}\|]$. Consider the symmballs (and coverands) $\widetilde{B}[\mathbf{a}_l, \frac{1}{2}]$, where $l = (l_1, l_2, \ldots, l_m)$, $l_j = \overline{0, N}$, and $a_{l,j} = -1 - \|\mathbf{v}\| + \frac{2}{N}\big(1 + \|\mathbf{v}\|\big) \cdot l_j$. In other words, centers of these symmballs are points of m-dimensional grid with the step $\gamma = \frac{2}{N}\big(1 + \|\mathbf{v}\|\big)$ along each axis of $\mathbb{R}^m$. There are $(N+1)^m$ such symmballs. Since Cor. 4, on the other hand, implies $\widetilde{B}[\mathbf{a}_l, \frac{1}{2}] \supseteq B[\mathbf{a}_l, \frac{1}{2}(1 - \|\mathbf{v}\|)]$, we seek N such that $B[\mathbf{0}, 1 + \|\mathbf{v}\|] \subseteq \bigcup\limits_{l} B[\mathbf{a}_l, \frac{1}{2}(1 - \|\mathbf{v}\|)]$ (that is, small balls are placed "densely enough").

$B[\mathbf{0}, 1 + \|\mathbf{v}\|]$ is a subset of the hypercube $H = \{\mathbf{x} \in \mathbb{R}^m \colon |x_j| \leqslant 1 + \|\mathbf{v}\|, j = \overline{1, m}\} = \bigcup\limits_{l\colon l_j < N, j = \overline{1,m}} H_l$, where $H_l = \big\{\mathbf{x} \in \mathbb{R}^m \colon a_{l,j} \leqslant x_j \leqslant a_{l,j} + \gamma, j = \overline{1, m}\big\}$ are small hypercubes between $\mathbf{a}_l$. It is clear that the point $\mathbf{h}_l \in H_l$ that is farthest from all its vertices $\{\mathbf{a}_{l'}\}$ is its center, $h_{l,j} = a_{l,j} + \frac{1}{2}\gamma$. $\|\mathbf{h}_l - \mathbf{a}_{l'}\| \equiv \big(\sum\limits_{j=1}^{m} |\frac{1}{2}\gamma|^p\big)^{\frac{1}{p}} = \frac{1}{2}\gamma m^{\frac{1}{p}}$.

Hence we need $\|\mathbf{h}_l - \mathbf{a}_{l'}\| \leqslant \frac{1}{2}(1 - \|\mathbf{v}\|) \Leftrightarrow \frac{2}{N}\big(1 + \|\mathbf{v}\|\big) m^{\frac{1}{p}} \leqslant 1 - \|\mathbf{v}\|$; the smallest N that satisfies it is $N = \lceil 2 \cdot \frac{1 + \|\mathbf{v}\|}{1 - \|\mathbf{v}\|} \cdot m^{\frac{1}{p}} \rceil$.

The number of coverands can be reduced. Take any $\mathbf{x} \in C[\mathbf{a}_l, \frac{1}{2}]$. $\|\mathbf{x}\| \geqslant \big|\|\mathbf{a}_l\| - \|\mathbf{x} - \mathbf{a}_l\|\big|$ and $\|\mathbf{x} - \mathbf{a}_l\| \leqslant \frac{1}{2}(1 + \|\mathbf{v}\|)$. Therefore, if $\|\mathbf{a}_l\| > \frac{3}{2}(1 + \|\mathbf{v}\|)$, then $\|\mathbf{x}\| > 1 + \|\mathbf{v}\|$, so $\mathbf{x} \notin C[\mathbf{0}, 1]$, — coverands whose anchors are such $\mathbf{a}_l$ can be discarded immediately.

$C_0 = C[\mathbf{a}_0, r]$ for ♣4 is a "big enough" symmball that contains $\mathbf{s}$. S for ♣5 is $\widetilde{B}[\mathbf{a}_0, 2r]$.

Our implementation of RCD algorithm in Rust (`https://rust-lang.org`) is available at `https://crates.io/crates/sr-rcd`. Here only 2 source files are reproduced: `rcd.rs` (the algorithm) and `random.rs` (its usage to solve random SRP). Note that Steps 1 and 3 are parallelized.

`rcd.rs`

```
extern crate rayon;
use rayon::prelude::*;
use crate::{point::Point, qmspace::QMSpace};
```

```
pub struct Sensor {
    p: Point,  t: f64
}

impl Sensor {
    pub fn make(p: &Point, t: f64) -> Self {
        Self{p: p.clone(), t}
    }
}

fn defect(x: &Point, sensorium: &Vec<Sensor>, qmsp: &impl QMSpace) -> f64 {
    let mut tau_min = f64::MAX;  let mut tau_max = f64::MIN;
    sensorium.iter().for_each(|r| {
        let tau = r.t - qmsp.tau(x, & r.p);
        tau_min = tau_min.min(tau);  tau_max = tau_max.max(tau);
    });
    tau_max - tau_min
}

pub fn rcd(sensorium: &Vec<Sensor>, qmsp: &impl QMSpace, anchor0: &Point, radius0: f64,
  precision: f64) -> Result<Point, String> {
    // Step 0
    let mut k: usize = 0;  let mut r: f64 = radius0;
    let mut cover_anchors = vec![anchor0.clone()];  let mut z: Point;
    loop {
        // Step 1
        let rc_anchors = cover_anchors.par_iter().map(|a| qmsp.make_rc_anchors(a, r)
            .into_par_iter()).flatten().collect::<Vec<Point>>();
        cover_anchors = rc_anchors.into_par_iter()
            .filter(|a| defect(a, sensorium, qmsp) <= r).collect::<Vec<Point>>();
        if cover_anchors.len() == 0 {
            break Err(format!("cover became empty"));
        }
        // Step 2
        z = cover_anchors[cover_anchors.len() >> 1].clone();
        // Step 3
        k += 1;  r *= 0.5;
        let d = r + cover_anchors.par_iter()
            .fold(|| 0f64, |max, a| max.max(qmsp.tau(&z, &a)).max(qmsp.tau(&a, &z)))
            .reduce(|| 0f64, |a, b| a.max(b));
        println!("Iteration {}: {} coverands, d = {:.4}", k, cover_anchors.len(), d);
        if d < precision {
            break Ok(z);
        }
    }
}
```

random.rs

```
extern crate rand;  extern crate rand_distr;  extern crate sr_rcd;
use {rand::prelude::*,  rand_distr::StandardNormal,  std::time::Instant,
     sr_rcd::{Point, QMSpace, rcd, RMP, Sensor}};

fn make_random_sr_problem(qmsp: &impl QMSpace, rad: f64, n: usize, err_dev: f64)
  -> (Vec<Sensor>, Point) {
    let mut rng = thread_rng();
    let s = Point::new_random(&mut rng, qmsp.dim(), rad);
    let t0 = rng.gen_range((-16.0)..=16.0);
    ((0..n).map(|_| {
```

```
        let p = Point::new_random(&mut rng, qmsp.dim(), rad);
        let err = err_dev * rng.sample::<f64, _>(StandardNormal);
        let t = t0 + qmsp.tau(&s, &p) + err;  Sensor::make(&p, t)
    }).collect::<Vec<Sensor>>(), s)
}

fn main() {
    let space = RMP::new(2, 3.14, 1e-4, &Point::new(vec![0.2, -0.15]));
    let (sensorium, s) = make_random_sr_problem(&space, 50.0, 32, 0.0);
    let start = Instant::now();
    match rcd(&sensorium, &space, &Point::new_default(space.dim()), 100.0, 0.1) {
        Ok(z) => {
            println!("Time: {:.3} sec", start.elapsed().as_secs_f64());
            println!("RCD-approximated source: {:.4?}", &z);
            println!("True source: {:.4?}", &s);
            println!("Quasi-metric errors: {:.4}, {:.4}", space.tau(&z, &s),
                space.tau(&s, &z));
        },
        Err(s) => println!("ERROR: {}", s)
    }
}
```

That is, the space is $\mathbb{R}^2_{3.14}$, accuracy of τ calculation is $\Delta = 10^{-4}$, and the wind is $\mathbf{v} = (0.2, -0.15)$. There are 32 sensors, which, along with the source, are placed randomly into $[-50; 50]^2$. The precision of source approximation is $\frac{1}{10}$.

We have observed 3 types of execution results for this set of parameters, depending on the behaviour of the number n_k of coverands:

1) (50%) n_k stays relatively small, $\mathbf{z}$ is obtained:

```
Iteration 1: 8 coverands, d = 363.4598
Iteration 2: 4 coverands, d = 76.0051
Iteration 3: 8 coverands, d = 38.0025
Iteration 4: 16 coverands, d = 19.0013
Iteration 5: 16 coverands, d = 3.1250
Iteration 6: 16 coverands, d = 1.5625
Iteration 7: 16 coverands, d = 0.7813
Iteration 8: 16 coverands, d = 0.3906
Iteration 9: 16 coverands, d = 0.1953
Iteration 10: 16 coverands, d = 0.0977

Time: 0.137 sec
RCD-approximated source: [-0.2388, 15.8431]
True source: [-0.2930, 15.8426]
Quasi-metric errors: 0.0678, 0.0452
```

2) (40%) n_k increases exponentially in general, $\mathbf{z}$ is obtained:

```
Iteration 1: 6 coverands, d = 242.2815
Iteration 2: 37 coverands, d = 155.1942
Iteration 3: 18 coverands, d = 38.0025
Iteration 4: 24 coverands, d = 19.0013
Iteration 5: 30 coverands, d = 9.5006
Iteration 6: 54 coverands, d = 4.7503
Iteration 7: 138 coverands, d = 3.9691
Iteration 8: 192 coverands, d = 1.1876
Iteration 9: 492 coverands, d = 0.9923
Iteration 10: 768 coverands, d = 0.2854
Iteration 11: 1536 coverands, d = 0.1484
Iteration 12: 3840 coverands, d = 0.1240
Iteration 13: 5376 coverands, d = 0.0371

Time: 7.057 sec
RCD-approximated source: [-47.8776, -5.8815]
True source: [-47.8649, -5.8804]
Quasi-metric errors: 0.0106, 0.0159
```

3) (10%) n_k increases exponentially in general and becomes too large for processing (requires too much time or there is not enough memory), so $\mathbf{z}$ is not obtained:

```
Iteration 1: 5 coverands, d = 180.1942
Iteration 2: 30 coverands, d = 191.0863
Iteration 3: 134 coverands, d = 187.9454
Iteration 4: 330 coverands, d = 54.3688
Iteration 5: 736 coverands, d = 23.8858
Iteration 6: 1110 coverands, d = 13.5922
Iteration 7: 2460 coverands, d = 5.3459
Iteration 8: 5600 coverands, d = 2.6729
Iteration 9: 10160 coverands, d = 0.9472
Iteration 10: 29040 coverands, d = 0.4736
Iteration 11: 128400 coverands, d = 0.4248
^C
```

(more than 1 minute has passed)

Robustness. To simulate inaccuracy in measurements, there is σ = `err_dev` parameter of `make_random_sr_problem()` function, which adds N_{0,σ^2}-distributed error to t_i. For $\sigma \in [0.01; 0.02] = [\frac{\delta}{10}; \frac{\delta}{5}]$ the algorithm obtains $\mathbf{z}$ almost as for $\sigma = 0$, with slightly increased quasi-metric errors and rare failures when at some iteration the cover becomes empty. For larger σ the algorithm fails more and more often; set $\sigma = 0.05 = \frac{\delta}{2}$ and the probability of failure exceeds 50%.

(Im)practicality. As soon as $m = 3$, (3) occurs in nearly 90% of executions, — this implementation seems to be impractical in 3D space on general purpose computers of nowadays. By Prop. 6 that excludes $n_k \xrightarrow[k\to\infty]{} \infty$, in fact $k \to \infty$ itself, (3) can be prevented by using a fast enough computer with large enough yet finite memory. There may be other optimizations.

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